General Relativity: Tutorial 3

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1. A particle moves with variable velocity $\mathbf{v}$ relative to some inertial frame, under the action of a force $\mathbf{f}$. Show that

$$\mathbf{f} = \frac{m_0}{\left[1 - \frac{v^2}{c^2}\right]^{1/2}} \frac{d\mathbf{v}}{dt} + \frac{m_0}{c^2 \left[1 - \frac{v^2}{c^2}\right]^{3/2}} v \frac{dv}{dt} \mathbf{v} \ ,$$

where v is the magnitude of $\mathbf{v}$. Infer that

$$\mathbf{f} = \frac{m_0}{\left[1 - \frac{v^2}{c^2}\right]^{3/2}} \frac{d\mathbf{v}}{dt}$$

if the acceleration is parallel to $\mathbf{v}$, while

$$\mathbf{f} = \frac{m_0}{\left[1 - \frac{v^2}{c^2}\right]^{1/2}} \frac{d\mathbf{v}}{dt}$$

if the acceleration is perpendicular to $\mathbf{v}$.

Suppose the particle moves along the x - axis under a force of magnitude

$$\mathbf{f} = \frac{2m_0 c^2 a}{(a - x)^2} \ ,$$

being at rest at $t = 0$. Show that the time taken to move to the point with coordinate x $(< a)$ is

$$t = \frac{1}{3c} \left[\frac{x}{a}\right]^{1/2} (x + 3a) \ .$$

2. By defining the number flux as $N^a = nU^a$, show by considering the flow of particles across the sides of a cube of length a, that conservation of particle number is given by ${N^a}_{,a} = 0$.

3. Show that Maxwells equations can be written as

$${F^{\alpha\beta}}_{,\beta} = 4\pi J^\alpha \ , \quad F_{\alpha\beta,\mu} + F_{\beta\mu,\alpha} + F_{\mu\alpha,\beta} = 0 \ ,$$

where $F^{\alpha\beta}$ is the Maxwell Tensor and J^α is the 4-current. You can take units where $c = 1$.

4. Using Maxwells equations show that ${J^\mu}_{,\mu} = 0$ and explain the physical meaning of this equation.

5. By writing $\mathbf{J} = q\mathbf{U}$ and using Maxwell's equations, show that

$$K^\mu = -T^{\mu\nu}{}_{,\nu} \ ,$$

where

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

is the energy - momentum tensor of the electromagnetic field. and $K^\mu = qF^{\mu\nu}U_\nu$ is the Lorentz force. Infer that the energy density is

$$T^{00} = \frac{1}{8\pi}\left(\mathbf{E}^2 + \mathbf{B}^2\right) \ .$$

Calculate the Poynting vector T^{0i} in terms of $\mathbf{E}$ and $\mathbf{B}$.

NOTE: This question is worth a bottle of wine!!!

6. An electrified particle having charge q and rest mass m_0 moves in a uniform electric field E in the x direction. If it is initially at rest at the origin, show that it moves along the x-axis so that at time t

$$x = \frac{1}{k}\left[\sqrt{1 + k^2t^2} - 1\right] \ ,$$

where $k = qE/m_0$ and we have taken units where $c = 1$.